10.022 Modelling Uncertainty, 2024

1D PROJECT

Due: 6:00pm, Friday 15 November

Problem statement: Consider the following simplistic model of an investment (which could apply to, say, the stock market). At each time step (say, each day), the value of your investment either goes up by 1 unit (say, \$1000) with probability p, or it goes down by 1 unit with probability $1 - p$.

To be more precise, let p be a real number strictly between 0 and 1, let $q = 1 - p$, and let n be a positive integer. For each $i \in \{1, 2, \ldots, n\}$, define a random variable X_i using the probability mass function

$$\mathbb{P}(X_i = 1) = p, \qquad \mathbb{P}(X_i = -1) = q.$$

Suppose that the X_i's are independent, so for instance $\mathbb{P}\big((X_1 = 1) \cap (X_2 = 1) \cap (X_3 = -1)\big) = p^2 q$, etc. Finally, define

$$S_n = \sum_{i=1}^{n} X_i.$$

Hence, after n time steps, your investment is worth $k + S_n$ units, where the non-negative integer k is the initial value of your investment.

The main goal of this project is to find the probability that the investment will reach a target value, or that it will reach the value 0.

Question 1 (warm up)

(a) What are the largest and smallest possible values of S_n, and what are the respective probabilities of getting these values?

(b) Find $\mathbb{P}(S_n = 0)$ as a function of n. (Hints: the answer depends on whether n is even or odd; use enumeration.)

(c) Find $\mathbb{P}(S_n = 2\,m + 1)$, where m is your *group number*, and $n \geq 2\,m + 1$.

(d) Write X_i as a linear function of a Bernoulli random variable, and hence, write S_n in terms of a suitable binomial random variable.

(e) Using (d), compute $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$.

(f) Using (d), approximate S_n by a normal random variable when n is large. (Hint: this uses Week 8 knowledge.)

Question 2 (analyzing the investment)

Fix a positive integer $N > k$. Recall that your investment is worth $k + S_n$ after n time steps; suppose that this process continues (that is, the number of steps n increases indefinitely), until the investment is worth N or 0. If your investment reaches N before it drops to 0 , then you are said to *win* (e. g. you have reached a target value). If your investment drops to 0 before it reaches N, then you are said to *lose* (e. g. you go bankrupt).

(a) Suppose that $k = 2$ and $N = 3$. Compute the probability that you win, as a function of p. (Hints: a picture might help; use an infinite sum.)

(b) For general k and N, finding the probability that you win is tricky, and we need to follow the approach given in the rest of this question. Define P_k to be the probability that you win, starting with an initial value of k. Using the law of total probability, show that, for any integer k where $1 \leq k \leq N - 1$,

$$P_k = p\,P_{k+1} + q\,P_{k-1}.$$

(Hint: this is conceptually similar to the second solution of Week 4 Class 1 Activity 5.)

(c) Then, show that the conclusion of (b) can be written as

$$P_{k+1} - P_k = \frac{q}{p}(P_k - P_{k-1}).$$

By varying k, show that all of the equations below are true:

$$P_2 - P_1 = \left(\frac{q}{p}\right) P_1, \qquad P_3 - P_2 = \left(\frac{q}{p}\right)^2 P_1, \qquad \ldots, \qquad P_k - P_{k-1} = \left(\frac{q}{p}\right)^{k-1} P_1.$$

(d) Now assume that $p \neq 1/2$. Sum the equations in (c), then show that

$$P_k = \frac{1 - (q/p)^k}{1 - q/p} P_1.$$

Also, explain why the assumption $p \neq 1/2$ is needed here.

(e) Finally, argue that $P_N = 1$, then find P_1, and hence find P_k for $p \neq 1/2$. Check that your expression for P_k gives the correct answer for (a).

(f) Now suppose that $p = 1/2$; find an expression for P_k. Give an intuitive explanation for this expression (this is open-ended).

(g, harder) Find the probability that you lose. Hence, find the probability that you never win or lose (that is, for any n, your investment's value is always strictly between 0 and N).

Question 3 (various scenarios)

Hint: for all the parts here, you should use the findings from Question 2.

(a) Suppose that $k =$ your group number, and $N = 3k$. What should p be such that you have a 50% chance of winning? Repeat this for $k = (10 \times$ your group number) and $N = 3k$. If p is considered to be the investor's skill level, then interpret your findings in terms of the skill required to triple your wealth.

(b) You and an opponent play a game consisting of many (independent) rounds; the winner of each round wins \$1 from the other player. You keep playing rounds until one player goes bankrupt (at which point the other player wins the game). Suppose that you are twice as skillful as your opponent (so, you have a 2/3 chance of winning each round, while your opponent has a 1/3 chance), but your opponent is twice as wealthy as you (so, you start with \$$k$, but your opponent starts with \$$(2k)$). Who is more likely to win the game?

(c) Box A contains 6 marbles and box B contains m marbles, where m is your group number. One of the boxes is selected at random (each box has a 50% chance), and one marble from the selected box is transferred to the other box. If this process is repeated indefinitely, then what is the probability that box A will become empty before box B becomes empty?

(d) Suppose that $k = 2$, $N = 7$, and $p = 1/2$. What is the probability that your investment reaches the value 6, before you lose?

(e) Suppose that $p < 1/2$. Show that, no matter how large the initial value k is, the probability of losing approaches 1 as $N \to \infty$. Give an interpretation of this result in terms of the investment.

(f) A drunk statistician stands next to a cliff, where taking 1 unit step directly towards the cliff would send him over the edge. With probability $1 - p$, he takes 1 unit step directly towards the cliff; with probability p, he takes 1 unit step directly away from the cliff. (In this idealized scenario, he continues the above behaviour for as long as possible.) What is the probability that he never falls over the edge? Here, define p as follows: find the mean of the student ID's of all your group members, subtract 10^6 from the mean, then divide the result by 10^4.

Instructions

Project groups

- Please work in your 2D project groups.

  If you are not in a 2D group for whatever reason (such as not taking other freshmore subjects), then please email the course leads *immediately*.

- Please think about how to divide up the work among the group members as early as possible.
- If someone in your group is not contributing despite your reminders and encouragement, then please let your instructors know as soon as you can.
- If you have questions about the *wording* of any parts of the project, then please ask your instructors (instead of asking students from another group).

Academic honesty

- Each group **must** work independently, without collaborating or interacting with any other groups. **Never** share your solutions with another group. Each group **must** write their own solutions. (This is already a group project, so there is no reason to work with other groups.)
- Several of the questions are open-ended, so if different groups produce similar solutions, then that will be regarded as highly suspicious, and can result in a 0 for *all groups involved*.
- You may consult books and online resources, but you **must** cite them in your work (in a reference section at the end of your report).

  Likewise, if you use methods and ideas outside the scope of the course, then briefly describe what they do, justify why they are needed, name any software used, and cite all references.

Submission

- Submission will be via eDimension; a link will be set up. Late submissions will incur a deduction.

  Any further instructions and updates will also be uploaded on eDimension.

- Each group should submit a **typed report in PDF format**, containing your solutions to the three questions. The page limit is four A4 sized pages (not including any cover page or references).
- The report should give the name of each group member, and a one-sentence description of how each member contributed to the project. Do *not* just say 'everyone contributed equally' (or something similar).
- Numerical answers should be expressed as simplified fractions, or rounded to 3 significant figures, whichever is more appropriate.

Grading

- The project will be graded on:

  - Correctness, clarity, and quality of your solutions,
  - Appropriate level of working and explanation shown,
  - Evidence of mastery of the course material, even creativity,
  - Presentation, readability, and appropriate citations.

- Your report should look much more professional than a typical homework submission. Please explain your logic (in words, not just with maths) for each solution, and **show all relevant steps**, so that a reader can easily follow and reconstruct your reasoning.